CS534 Introduction to Computer Vision

Linear Filters

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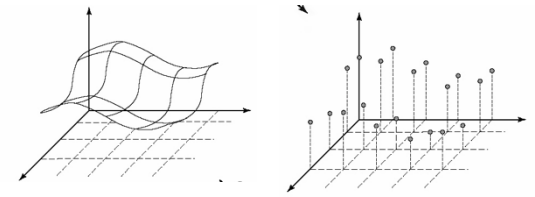
Outlines

- What are Filters
- Linear Filters
- Convolution operation
- Properties of Linear Filters
- Application of filters
- Nonlinear Filter
- Normalized Correlation and finding patterns in images
- Sources:
  - Forsyth and Ponce "Computer Vision a Modern approach" Chapter 4
  - Burger and Burge "Digital Image Processing" Chapter 6

Digital image

- Assume we use a gray-level image
- Digital image: a two-dimensional light intensity function f (x,y) where x and y denote spatial coordinates, the value of f at any point is proportional to the brightness (gray level) of the image at that point.
- A digital image:
  - is discretized in the spatial domain
  - Is discretized in the brightness domain.



- What operations can we perform on pixels?
  - Point operations
  - Filters

Point Operations

- Point Operations perform a mapping of the pixel values without changing the size, geometry, or local structure of the image
- Each new pixel value *I'(u,v)* depends on the previous value *I(u,v)* at the same position and on a mapping function *f*(.)
- The function *f*(.) is independent of the coordinates
- Such operation is called "homogeneous point operations"

$$a' \leftarrow f(a)$$
$$I'(u,v) \leftarrow f\big(I(u,v)\big)$$

Example of homogeneous point operations:

- Modifying image brightness or contrast
- Applying arbitrary intensity transformation (curves)
- Quantizing (posterizing) images
- Global thresholding
- Gamma correction
- Color transformations

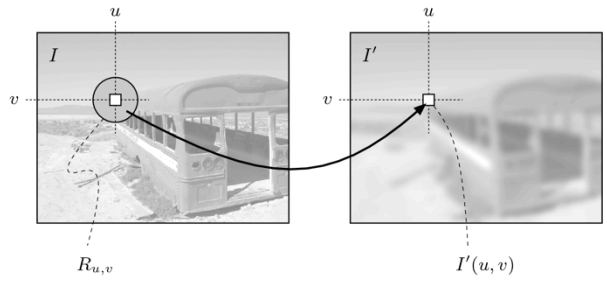
What is a Filter

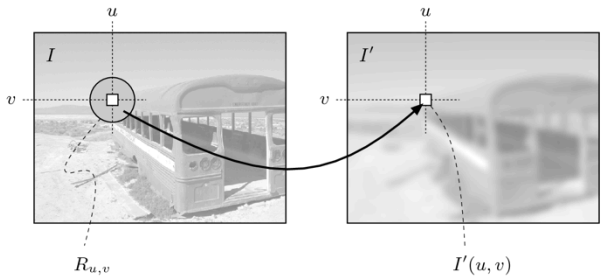
- Point operations are limited (why)
- They cannot accomplish tasks like sharpening or smoothing,
- We need a function that involves the intensities (color) in the neighborhood of each pixel

Smoothing an image by averaging

- Replace each pixel by the average of its neighboring pixels
- Assume a 3x3 neighborhood:

$$I'(u,v) \leftarrow \frac{p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8}{9}$$

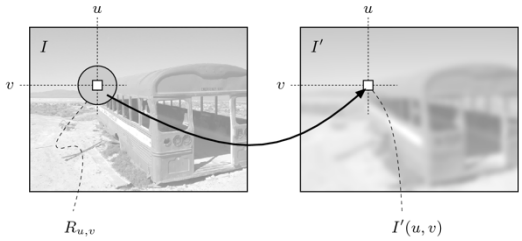




$$I'(u,v) \leftarrow \frac{p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8}{9}$$

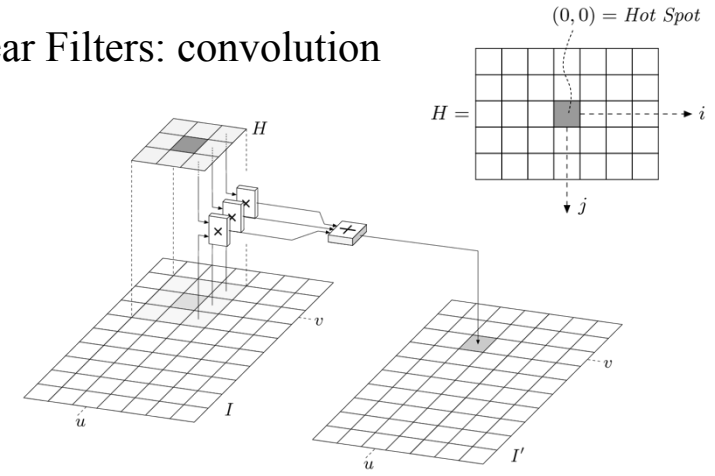
$$\begin{aligned} I'(u,v) \leftarrow \tfrac{1}{9} \cdot [\, & I(u-1,v-1) + I(u,v-1) + I(u+1,v-1) + \\ & I(u-1,v) + I(u,v) + I(u+1,v) + \\ & I(u-1,v+1) + I(u,v+1) + I(u+1,v+1) \,] \end{aligned}$$

$$I'(u,v) \leftarrow \frac{1}{9} \cdot \sum_{j=-1}^{1} \sum_{i=-1}^{1} I(u+i, v+j)$$



- In general a filter applies a function over the values of a small neighborhood of pixels to compute the result
- The size of the filter = the size of the neighborhood: 3x3, 5x5, 7x7, …, 21x21,..
- The shape of the filter region is not necessarily square, can be a rectangle, a circle…
- Filters can be linear of nonlinear

Linear Filters: convolution



$$I'(u,v) \leftarrow \sum_{(i,j)\in R_H} I(u+i, v+j) \cdot H(i,j)$$

$$I'(u,v) \leftarrow \sum_{i=-1}^{i=1} \sum_{j=-1}^{j=1} I(u+i, v+j) \cdot H(i,j)$$

Averaging filter

$$I'(u,v) \leftarrow \frac{p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8}{9}$$

$$\begin{aligned} I'(u,v) \leftarrow \tfrac{1}{9} \cdot [\, & I(u-1,v-1) + I(u,v-1) + I(u+1,v-1) + \\ & I(u-1,v) + I(u,v) + I(u+1,v) + \\ & I(u-1,v+1) + I(u,v+1) + I(u+1,v+1) \,] \end{aligned}$$

$$H(i,j) = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

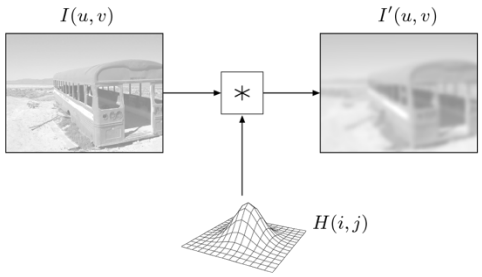
$$I'(u,v) \leftarrow \sum_{i=-1}^{i=1} \sum_{j=-1}^{j=1} I(u+i, v+j) \cdot H(i,j)$$

Mathematical Properties of Linear Convolution

- For any 2D discrete signal, convolution is defined as:

$$I'(u,v) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} I(u-i, v-j) \cdot H(i,j)$$

$$I' = I * H$$



Properties

$$I'(u,v) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} I(u-i, v-j) \cdot H(i,j)$$

- Commutativity

$$I * H = H * I$$

- Linearity

$$(s \cdot I) * H \;=\; I * (s \cdot H) \;=\; s \cdot (I * H)$$

$$(I_1 + I_2) * H \;=\; (I_1 * H) + (I_2 * H)$$

(notice) $(b + I) * H \;\neq\; b + (I * H)$

- Associativity

$$A * (B * C) = (A * B) * C$$

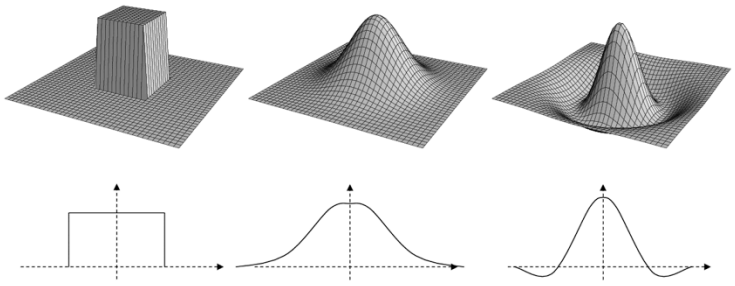
Properties

- Separability

$$H = H_1 * H_2 * \ldots * H_n$$

$$\begin{aligned} I * H &= I * (H_1 * H_2 * \ldots * H_n) \\ &= \big(\ldots((I * H_1) * H_2) * \ldots * H_n\big) \end{aligned}$$

Types of Linear Filters



0	0	0	0	0
0	1	1	1	0
0	1	1	1	0
0	1	1	1	0
0	0	0	0	0

(a)

0	1	2	1	0
1	3	5	3	1
2	5	9	5	2
1	3	5	3	1
0	1	2	1	0

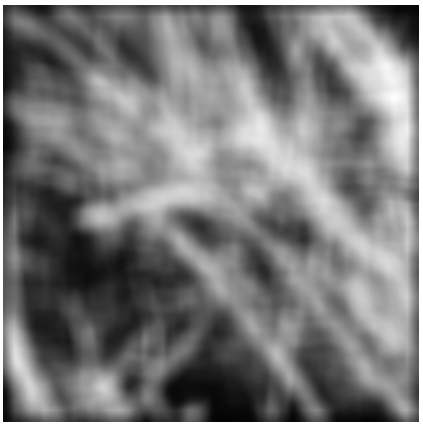
(b)

0	0	-1	0	0
0	-1	-2	-1	0
-1	-2	16	-2	-1
0	-1	-2	-1	0
0	0	-1	0	0

(c)

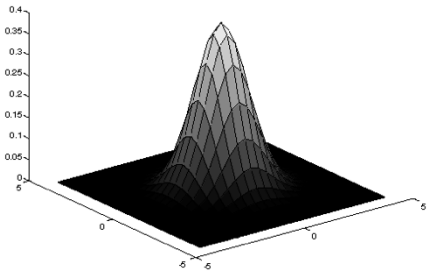
Smoothing by Averaging vs. Gaussian

Flat kernel: all weights equal 1/N



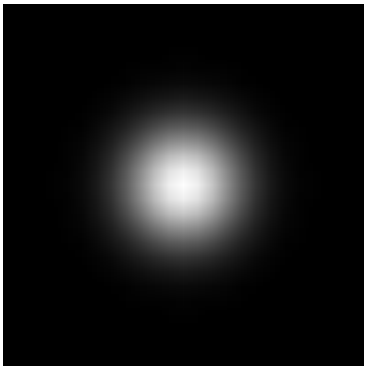
Smoothing with a Gaussian

- Smoothing with an average actually doesn't compare at all well with a defocussed lens
  - Most obvious difference is that a single point of light viewed in a defocussed lens looks like a fuzzy blob; but the averaging process would give a little square.



- A Gaussian gives a good model of a fuzzy blob

An Isotropic Gaussian

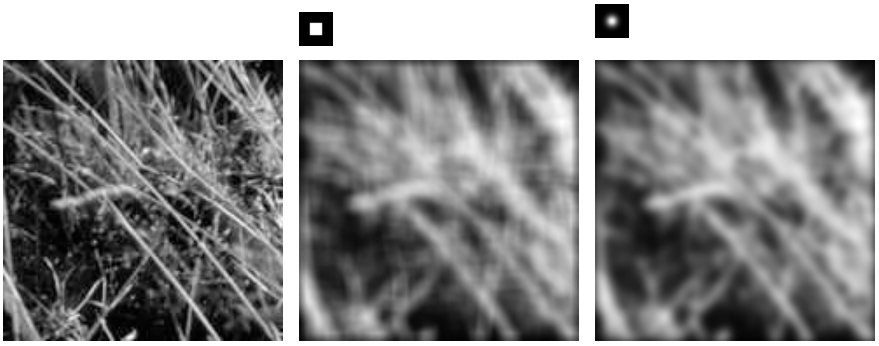


- The picture shows a smoothing kernel proportional to

$$\exp\left(-\left(\frac{x^2+y^2}{2\sigma^2}\right)\right)$$

(which is a reasonable model of a circularly symmetric fuzzy blob)

Smoothing with a Gaussian



Gaussian smoothing

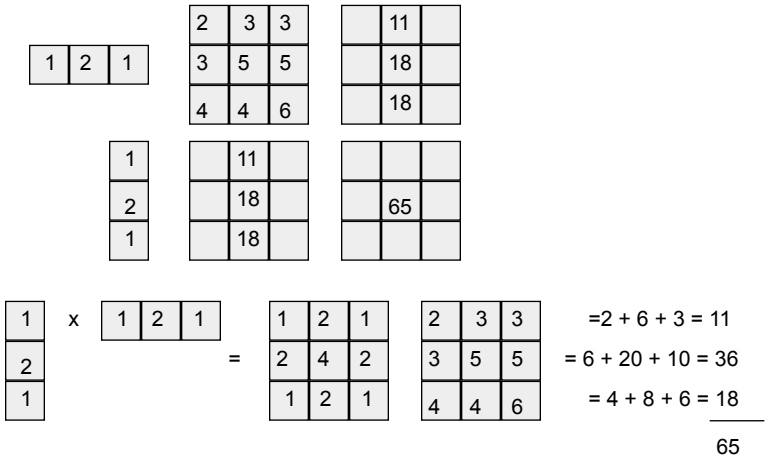
- Advantages of Gaussian filtering
  - rotationally symmetric (for large filters)
  - filter weights decrease monotonically from central peak, giving most weight to central pixels
  - Simple and intuitive relationship between size of σ and the smoothing.
  - The Gaussian is separable:

$$e^{-\frac{(x^2+y^2)}{2\sigma^2}} = e^{-\frac{x^2}{2\sigma^2}} * e^{-\frac{y^2}{2\sigma^2}}$$

Advantage of seperability

- First convolve the image with a one dimensional horizontal filter
- Then convolve the result of the first convolution with a one dimensional vertical filter
- For a *k*x*k* Gaussian filter, 2D convolution requires k^2 operations per pixel
- But using the separable filters, we reduce this to *2k* operations per pixel.

Separability



Advantages of Gaussians

- Convolution of a Gaussian with itself is another Gaussian
  - so we can first smooth an image with a small Gaussian
  - then, we convolve that smoothed image with another small Gaussian and the result is equivalent to smoother the original image with a larger Gaussian.
  - If we smooth an image with a Gaussian having sd σ twice, then we get the same result as smoothing the image with a Gaussian having standard deviation $(2\sigma)^{1/2}$

Noise

- Simplest noise model
  - independent stationary additive Gaussian noise
  - the noise value at each pixel is given by an independent draw from the same normal probability distribution

$$f_{observed}(x,y) = f(x,y) + N(0,\sigma^2)$$

- Issues
  - this model allows noise values that could be greater than maximum camera output or less than zero
  - for small standard deviations, this isn't too much of a problem - it's a fairly good model
  - independence may not be justified (e.g. damage to lens)
  - may not be stationary (e.g. thermal gradients in the ccd)

sigma=1

sigma=16

The response of a linear filter to noise

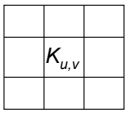
- Assume stationary independent additive Gaussian noise with zero mean (non-zero mean is easily dealt with)
- **Mean**:
  - output is a weighted sum of inputs
  - so we want mean of a weighted sum of zero mean normal random variables
  - must be zero
- **Variance**:
  - recall
    - variance of a sum of random variables is sum of their variances
    - variance of constant times random variable is constant^2 times variance
  - then if σ^2 is noise variance and kernel is K, variance of response is

$$f_{observed}(x,y) = f(x,y) + N(0,\sigma^2)$$

$$g * f_{observed} = g * f + g * N(0,\sigma^2)$$

$$N'(0,\sigma'^2)$$

$$\sigma^2 \sum_{u,v} K_{u,v}^2$$



The response of a linear filter to noise

$$f_{observed}(x,y) = f(x,y) + N(0,\sigma^2)$$

$$g * f_{observed} = g * f + g * N(0,\sigma^2)$$

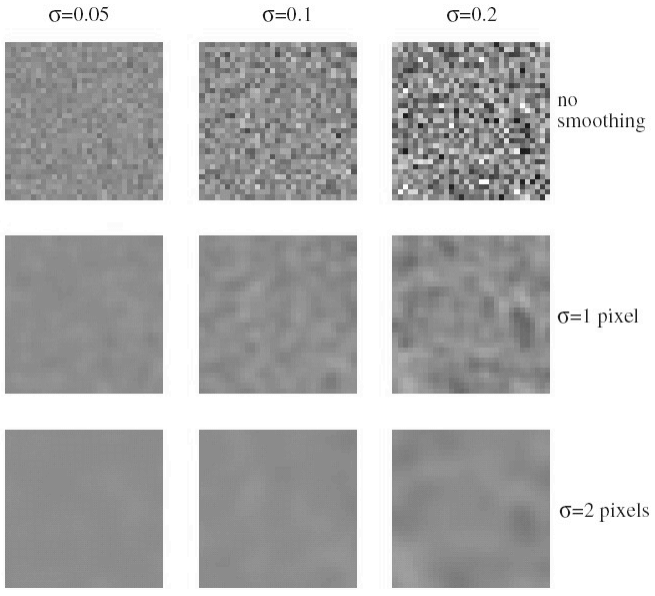
$$\Downarrow$$

$$N'(0,\sigma'^2) \qquad \sigma'^2 = \sigma^2 \sum_{u,v} K_{u,v}^2$$

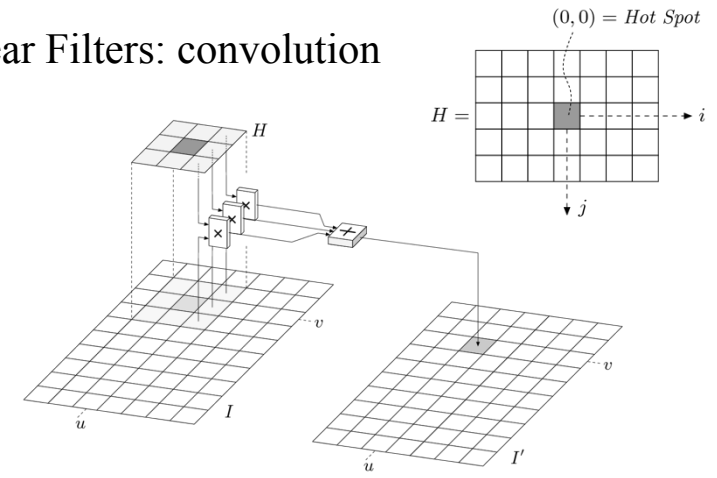
This can magnify or reduce the variance of the noise based on $\sum_{u,v} K_{u,v}^2$

If $\sum_{u,v} K_{u,v} = 1 \Rightarrow \sum_{u,v} K_{u,v}^2 \leq 1$ This reduces noise variance

(assume positive coefficients)



Linear Filters: convolution

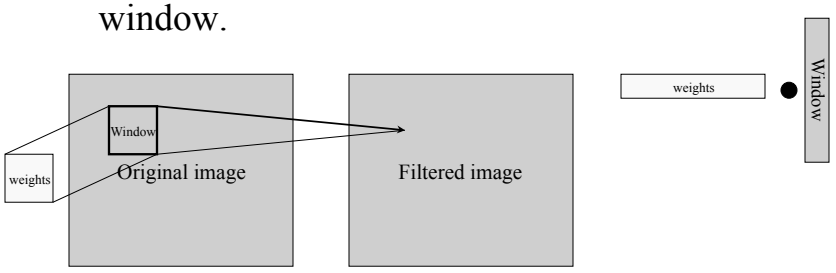


$$I'(u,v) \leftarrow \sum_{(i,j)\in R_H} I(u+i, v+j) \cdot H(i,j)$$

$$I'(u,v) \leftarrow \sum_{i=-1}^{i=1} \sum_{j=-1}^{j=1} I(u+i, v+j) \cdot H(i,j)$$

Convolution as a Dot Product

- Applying a filter at some point can be seen as taking a dot-product between the image and some vector
- Convoluting an image with a filter is equivalent to taking the dot product of the filter with each image window.

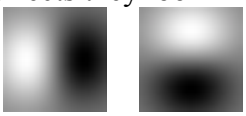


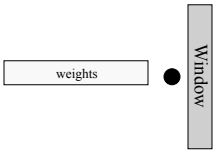
filters and finding patterns

- Largest value when the vector representing the image is parallel to the vector representing the filter
- Filter responds most strongly at image windows that looks like the filter.
- Filter responds stronger to brighter regions! (drawback)

Insight:

- filters look like the effects they are intended to find
- filters find effects they look like



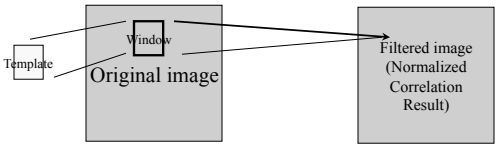


Ex: Derivative of Gaussian used in edge detection looks like edges

Normalized Correlation

- Convolution with a filter can be used to find templates in the image.
- Normalized correlation output is filter output, divided by root sum of squares of values over which filter lies
- Consider template (filter) *M* and image window *N*:

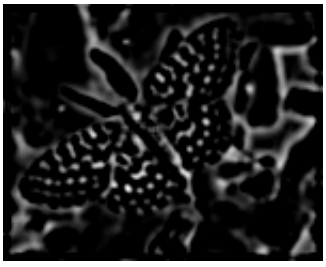
$$C = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} M(i,j)N(i,j)}{\left[\sum_{i=1}^{n}\sum_{j=1}^{n} M(i,j)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} N(i,j)^2\right]^{1/2}}$$



Normalized Correlation

$$C = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} M(i,j)N(i,j)}{\left[\sum_{i=1}^{n}\sum_{j=1}^{n} M(i,j)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} N(i,j)^2\right]^{1/2}}$$

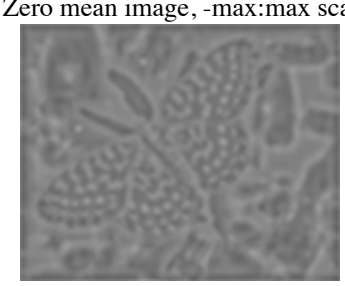
- This correlation measure takes on values in the range [0,1]
- it is 1 if and only if N = cM for some constant c
- so N can be uniformly brighter or darker than the template, M, and the correlation will still be high.
- The first term in the denominator, $\Sigma\Sigma M^2$ depends only on the template, and can be ignored
- The second term in the denominator, $\Sigma\Sigma N^2$ can be eliminated if we first normalize the grey levels of *N* so that their total value is the same as that of *M* - just scale each pixel in N by $\Sigma\Sigma M / \Sigma\Sigma N$



Positive responses

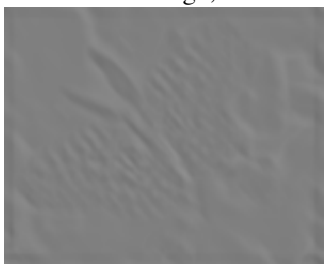
Zero mean image, -1:1 scale

Zero mean image, -max:max scale



Positive responses

Zero mean image, -1:1 scale



Zero mean image, -max:max scale

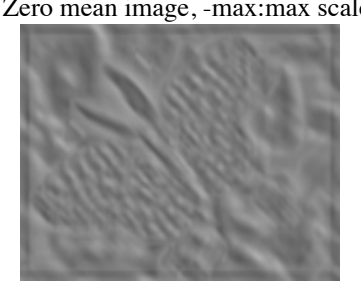
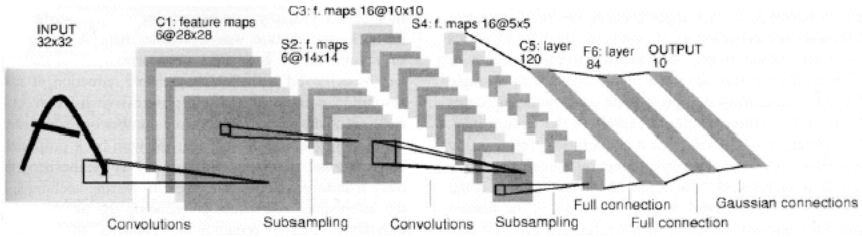


Figure from "Computer Vision for Interactive Computer Graphics," W.Freeman et al, IEEE Computer Graphics and Applications, 1998 copyright 1998, IEEE



Example of a biologically motivated recognition system
A convolutional neural network, LeNet; the layers filter, subsample, filter, subsample, and finally classify based on outputs of this process.

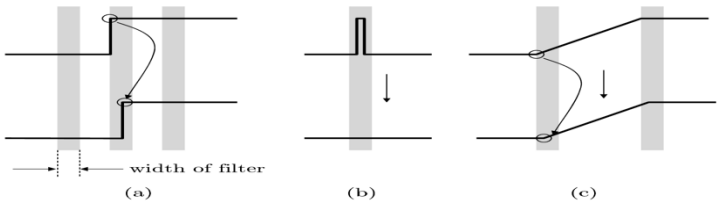
Figure from "Gradient-Based Learning Applied to Document Recognition", Y. Lecun et al Proc. IEEE, 1998 copyright 1998, IEEE

Nonlinear Filters

- Linear filters have a disadvantage when used for smoothing or removing noise: all image structures are blurred, the quality of the image is reduced.
- Examples of nonlinear filters:
  - Minimum and Maximum filters

$$I'(u,v) \leftarrow \min\{I(u+i,v+j) \mid (i,j) \in R\}$$

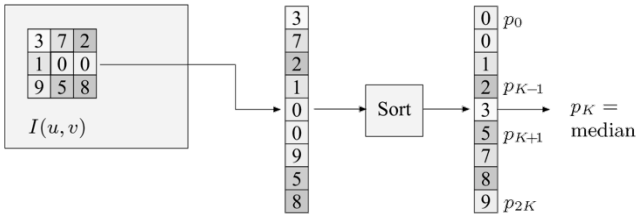
$$I'(u,v) \leftarrow \max\{I(u+i,v+j) \mid (i,j) \in R\}$$



Median Filter

- Much better in removing noise and keeping the structures

$$I'(u,v) \leftarrow \text{median}\,\{I(u+i, v+j) \mid (i,j) \in R\}$$



Weighted median filter